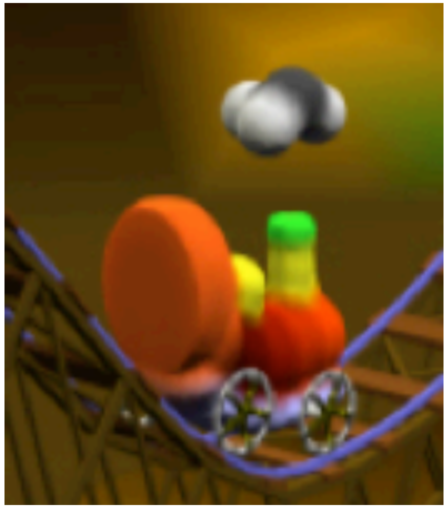
Bremen

Advanced Computer Graphics
Modelling beyond Polygons (and Raytracing them ...)



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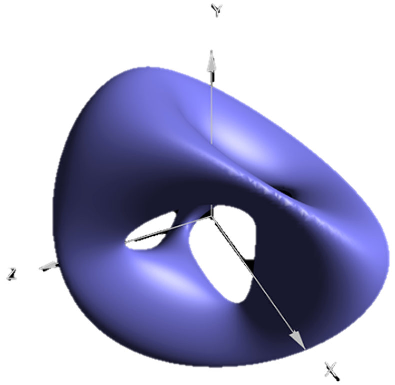
Implicit Surfaces

- An implicit surface is the set

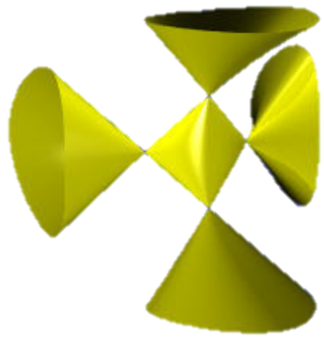
$$\{ \mathbf{x} \mid F(\mathbf{x}) = 0 \,,\ \mathbf{x} \in \mathbb{R}^3 \}$$

  with some function *F*.

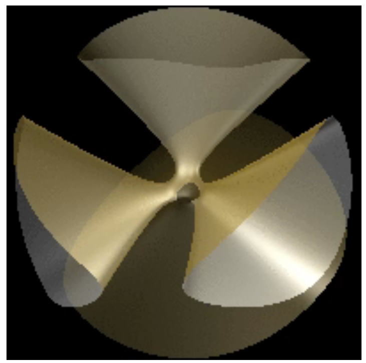
- Example: surface of sphere
- More & nicer examples:



$(x^2 + y^2 + z^2 - ak^2)^2 - b((z-k)^2 - 2x^2)\,((z+k)^2 - 2y^2)^2$



$8x^2 - xy^2 + xz^2 + y^2 + z^2 - 8$

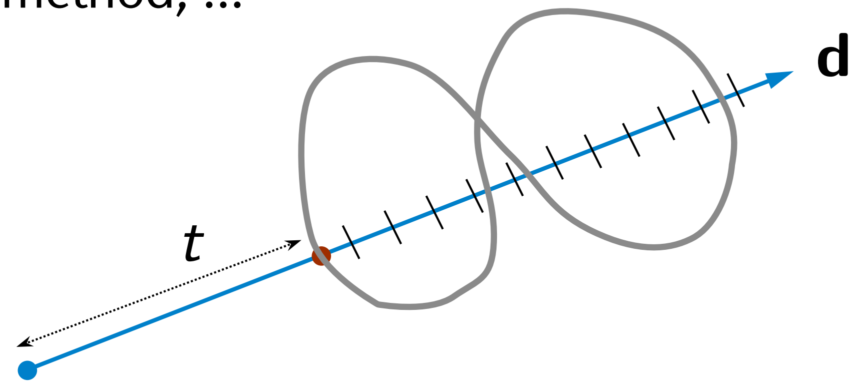


movie

$(x + y + z)^3 = x^3 + y^3 + z^3 + 1$

Intersection of Ray with Implicit Surface

- Ray: $P(t) = O + t \cdot \mathbf{d}$
- Inserting in implicit function $F(\mathbf{x}) = 0$
  yields polynomial
  $$F(\, P(t)\,) = 0$$
  in t of degree n
- Find the roots:
  - If degree < 5: solve for t analytically
  - Else: interval bisection, Newton's method, ...
  - Start values? ...



Root-Finding with Laguerre's Method

- Advantage: one of the very few "*sure-fire*" methods
- Limitations:
  - Works only for polynomials
  - Algorithm needs to perform calculations in complex numbers, even if all roots are real (and thus all coefficients)
  - Very little theory is known about its convergence behavior
    - If the root it converges to is a simple root, then the convergence order is (at least) 3
- Lots of empirical evidence that the algorithm (almost) always converges towards a root; and it does so from (almost) any starting value!

Motivation for the Algorithm

- Given: the polynomial

$$P(x) = (x - x_1)(x - x_2)\dots(x - x_n) \qquad (0)$$

where the x_i are the, possibly complex, yet unknown roots

- From that, we can derive the following equations:

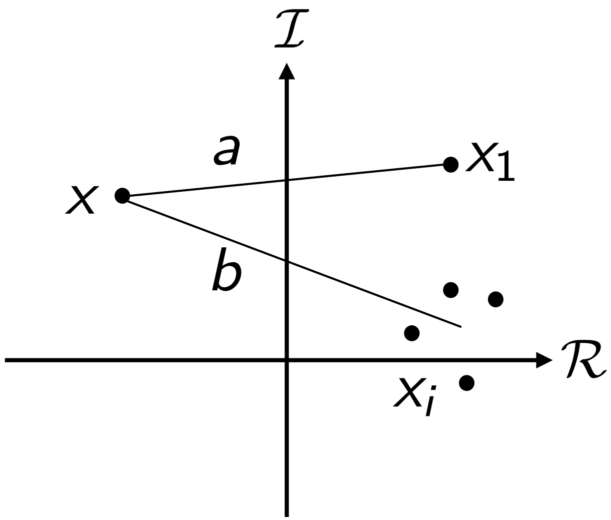
$$\ln|P(x)| = \ln|x - x_1| + \ln|x - x_2| + \dots + \ln|x - x_n|$$

$$\frac{d}{dx}\ln|P(x)| = \frac{1}{x - x_1} + \dots + \frac{1}{x - x_n} = \frac{P'(x)}{P(x)} =: G \qquad (1)$$

$$\frac{d^2}{dx^2}\ln|P(x)| = -\frac{1}{(x - x_1)^2} - \dots - \frac{1}{(x - x_n)^2}$$

$$= \frac{P''(x)}{P(x)} - \left(\frac{P'(x)}{P(x)}\right)^2 =: -H \qquad (2)$$

- Let x be our current approximation of a root, w.l.o.g. root x_1



- Make a "drastic" assumption:
  - Denote distance $x - x_1 = a$
  - Assume, distance to all other roots is

$$x - x_i \approx b, \quad i = 2, 3, \ldots, n$$

- Then, we can write (1) & (2) like this:

$$G \approx \frac{1}{a} + \frac{n-1}{b} \tag{3}$$

$$H \approx \frac{1}{a^2} + \frac{n-1}{b^2} \tag{4}$$

- Plug (4) into (3) and solve for a :

$$a \approx \frac{n}{G \pm \sqrt{(n-1)(nH - G^2)}} \qquad (5)$$

  - Compute G and H from

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$
$$P'(x) = a_1 + 2a_2 x + 3a_3 x^2 \ldots + n a_n x^{n-1}$$
$$P''(x) = 2a_2 + 3 \cdot 2 \cdot a_3 x \ldots + n \cdot (n-1) a_n x^{n-2}$$

  - Choose sign in front of sqrt such that $|a|$ becomes minimal
  - Remark: discriminant under sqrt can become negative $\longrightarrow a$ can become complex
- New approximation of root x_1 is $x_1 = x - a$

The Algorithm

```
choose 0-th approximation x^(0)
repeat
        compute  G = P'(x^(k)) / P(x^(k))

                 H = G^2 - P''(x^(k)) / P(x^(k))

        compute  a = n / (G ± sqrt((n-1)(nH - G^2)))

        let x^(k+1) = x^(k) - a
until a is "small enough" or k ≥ max
```

choose 0-th approximation $x^{(0)}$

repeat

compute $G = \frac{P'(x^{(k)})}{P(x^{(k)})}$

$H = G^2 - \frac{P''(x^{(k)})}{P(x^{(k)})}$

compute $a = \frac{n}{G \pm \sqrt{(n-1)\,(nH - G^2)}}$

let $x^{(k+1)} = x^{(k)} - a$

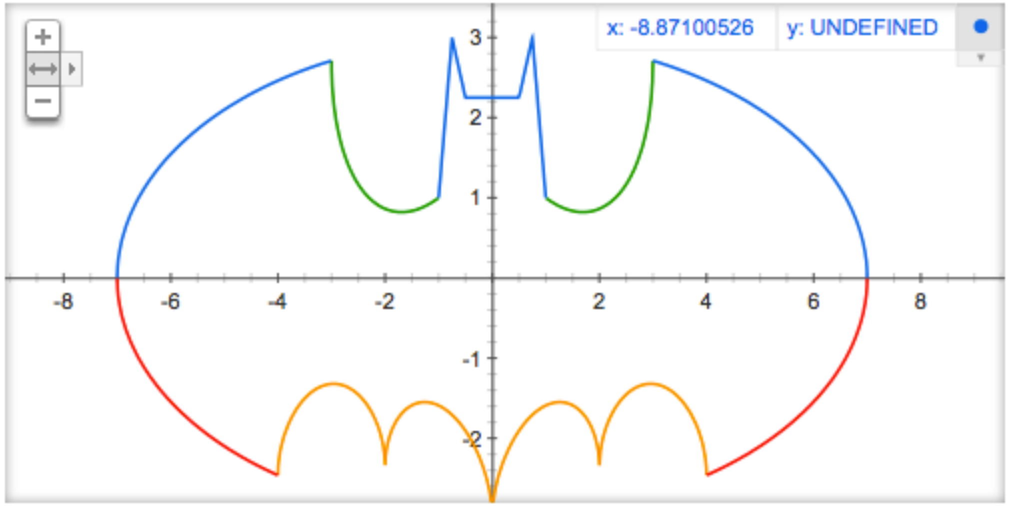
until *a is* "small enough" or $k \geq$ max

- Warning: try to use code from *Numerical Recipes*
- For ray-tracing: have to compute all roots!
  - When first root is found, factor it out of polynomial
  - Find next root of smaller polynomial, repeat Laguerre *n* times

The "Batman Equation"

Optional

- With a few tricks, one can even create complex objects 😀 :



$$\left(\left(\frac{x}{7}\right)^2 \sqrt{\frac{||x|-3|}{|x|-3}} + \left(\frac{y}{3}\right)^2 \sqrt{\frac{\left|y+\frac{3\sqrt{33}}{7}\right|}{y+\frac{3\sqrt{33}}{7}}} - 1\right) \cdot$$

$$\left(\left|\frac{x}{2}\right| - \left(\frac{3\sqrt{33}-7}{112}\right) x^2 - 3 + \sqrt{1-(||x|-2|-1)^2} - y\right) \cdot$$

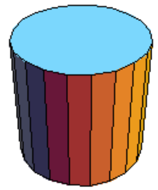
$$\left(9\sqrt{\frac{|(|x|-1)(|x|-.75)|}{(1-|x|)(|x|-.75)}}\right) \cdot \left(3|x| + .75\sqrt{\frac{|(|x|-.75)(|x|-.5)|}{(.75-|x|)(|x|-.5)}} - y\right) \cdot \left(2.25\sqrt{\frac{|(|x|-1)(|x|-.75)}{(1-|x|)(|x|-.75}}\right.$$

$$\left(\frac{6\sqrt{10}}{7} + (1.5-.5|x|)\sqrt{\frac{||x|-1|}{|x|-1}} - \frac{6\sqrt{10}}{14}\sqrt{4-(|x|-1)^2} - y\right) = 0$$

Quadrics (Sphere, Cylinder, Paraboloid, Hyperboloid)

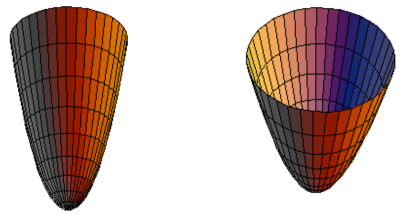
- Infinite cylinder:

$$x^2 + y^2 = 1$$



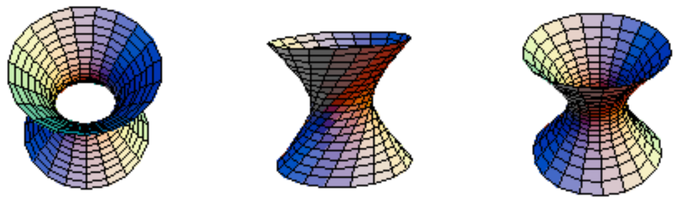
- Parabolloid:

$$x^2 + y^2 - z = 0$$



- Hyperboloid (*one sheet*):

$$x^2 + y^2 - z^2 = 1$$

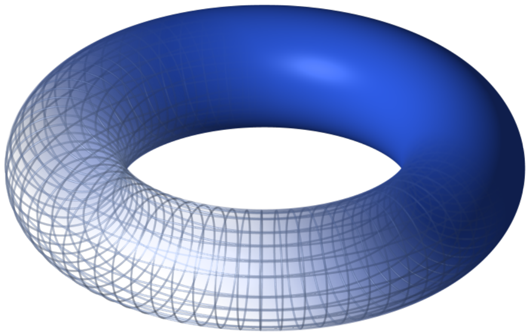


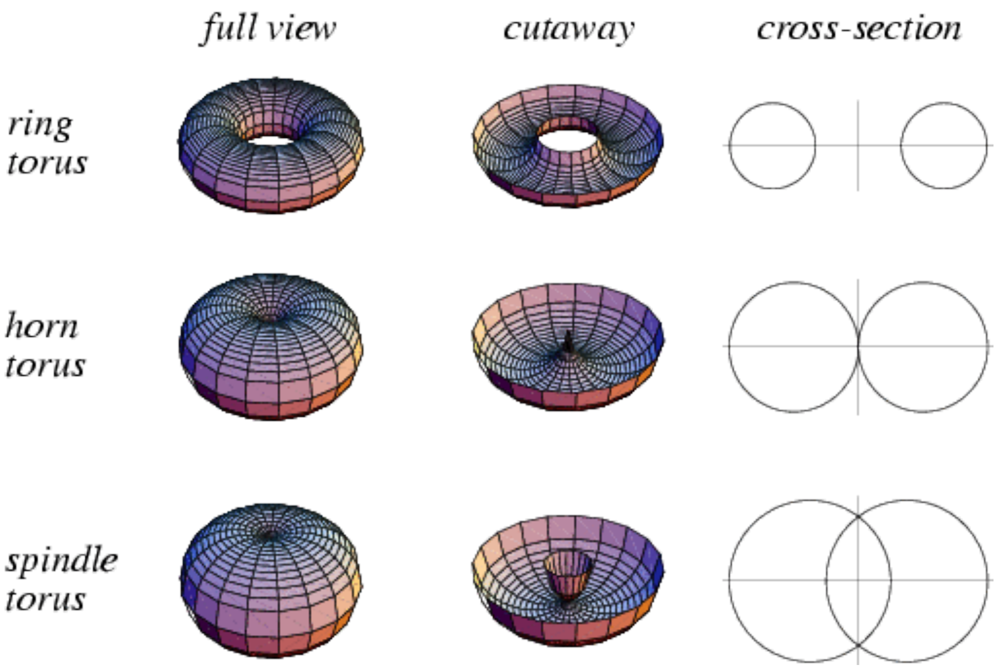
- All of these can be written as a quadratic form (hence the name):

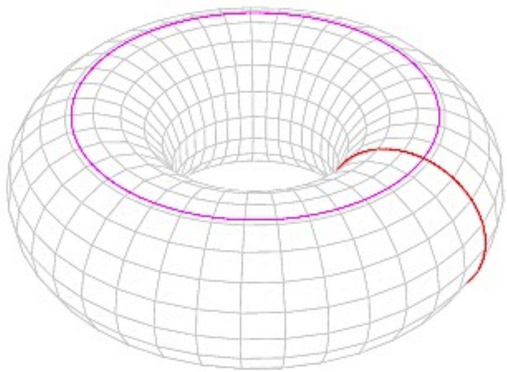
$$\mathbf{x}^T M \mathbf{x} = 0, \quad \mathbf{x} \in \mathbb{R}^4, \; M \in \mathbb{R}^{4\times 4}$$

- Torus (is not really a quadric!):

$$\left(c - \sqrt{x^2 + y^2}\right)^2 + z^2 = a^2$$





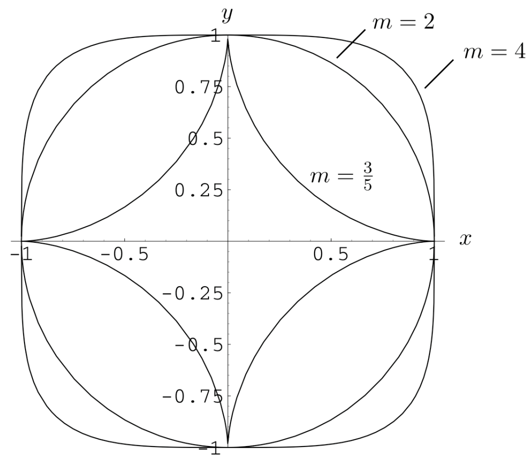


Superquadrics

Optional

- Generalization of quadrics
- Super-ellipsoid:

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$$



- Super-hyperboloid:

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q - \left(\frac{z}{c}\right)^r = 1$$

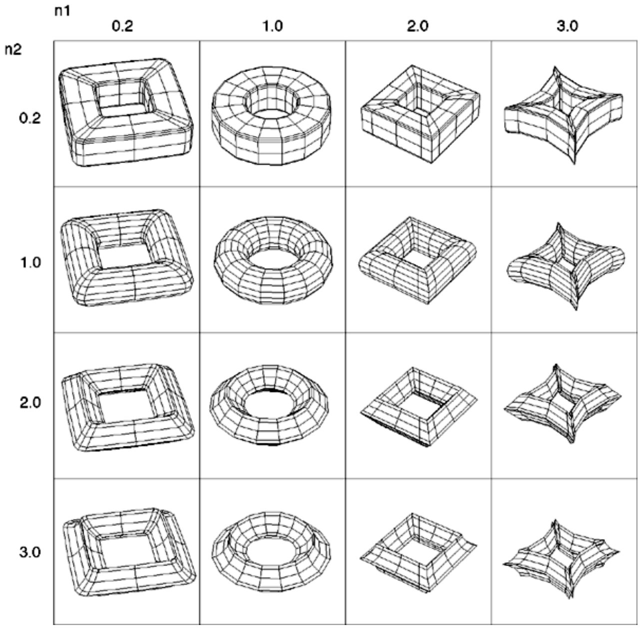
- Super-toroid:

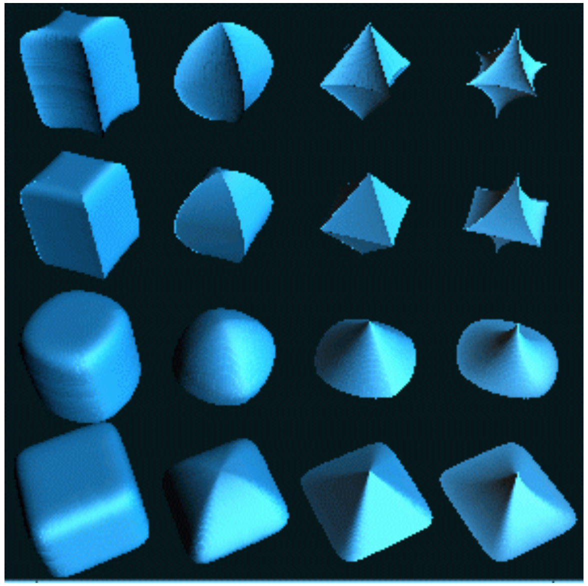
$$\left(d - \left(\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^n\right)^q\right)^r + \left(\frac{z}{c}\right)^p = e^2$$

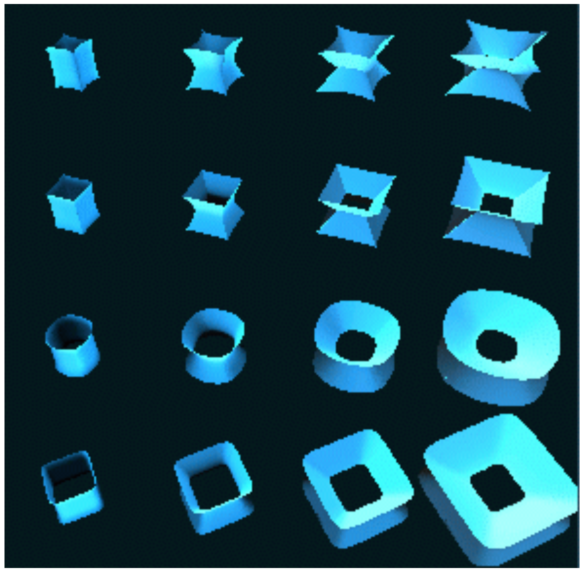
- Warning: in above equations, we always mean $|x|^p$!

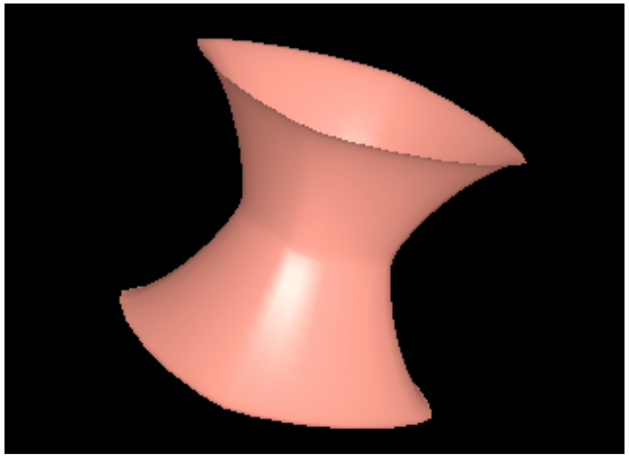
Examples of Super-Quadrics

Optional









XScreenSaver demo "SuperQuadrics"
(www.jwz.org/xscreensaver)

Ratioquadrics

Optional

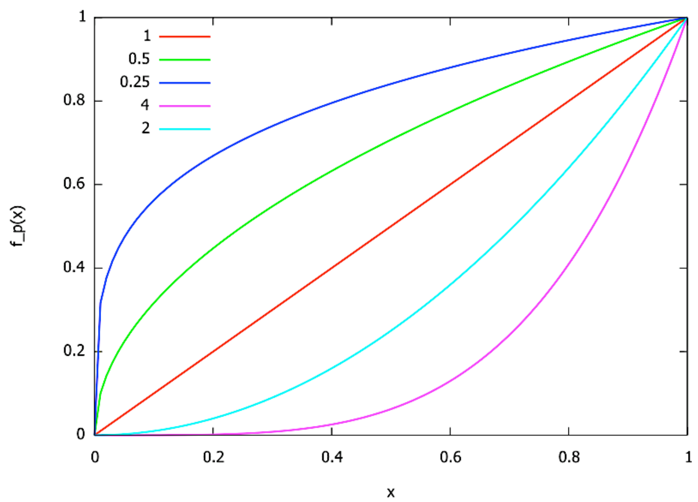
[Blanc & Schlick, 1996]

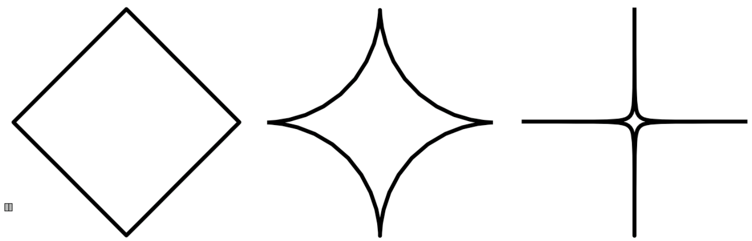
- Variant of superquadrics with somewhat better properties
- Idea of superquadrics can be rewritten like this:

$$F(x,y,z) = f_p(\tfrac{x}{a}) + f_q(\tfrac{y}{b}) + f_r(\tfrac{z}{c}) - 1$$

$$f_p(x) = |x|^p$$

- Problem:
  - $f_p(x)$ is not differentiable at x=0 for $p \leq 1$
  - Therefore, we get cusps, which might be unwanted
  - Besides, $f_p(x)$ is fairly expensive to evaluate

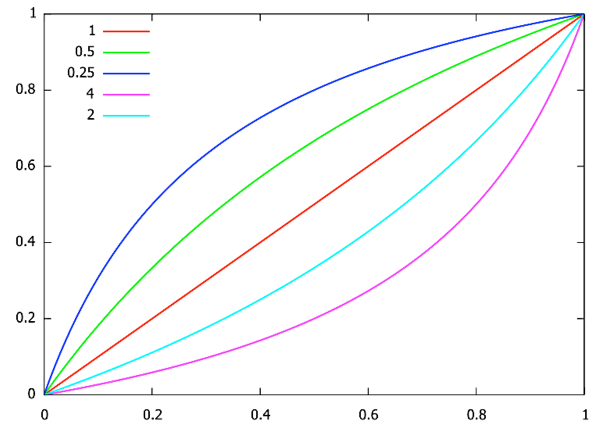




Optional

- Simple idea: use different power functions
- A new pseudo-power function [Blanc & Schlick]:

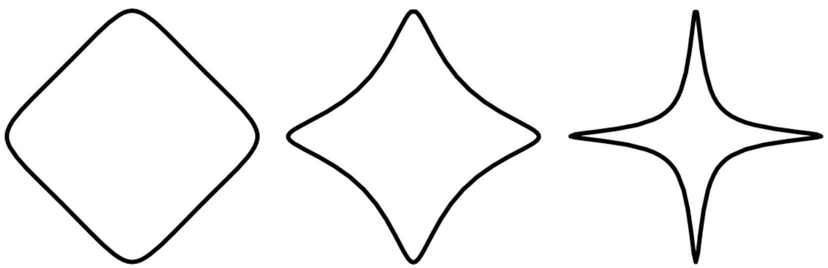
$$g_p(x) = \frac{x}{p + (1-p)x}$$



- With that, the ratioquadric for a "ratio-ellipsoid" is

$$F(x,y,z) = g_p\left(\tfrac{x}{a}\right) + g_q\left(\tfrac{y}{b}\right) + g_r\left(\tfrac{z}{c}\right) - 1$$

- Result:



Metaballs

- Inspired by molecules
- Idea: consider the surface of a sphere as the set of points that have the same "potential", where the maximum is reached at the center of the sphere → isosurface
- A potential field is described by a potential field function, e.g.

$$p(r) = \frac{1}{r^2}$$

where

$$r = r_1(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_1\|$$

- The sphere's surface is thus

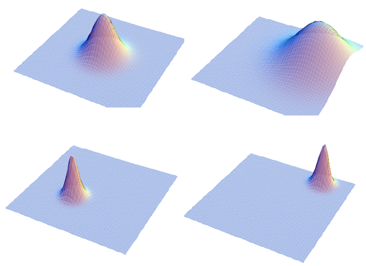
$$K = \{\mathbf{x} \mid p(\mathbf{x}) = \tau\}$$

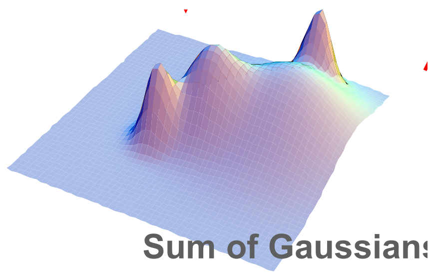
  - τ is called threshold or isovalue

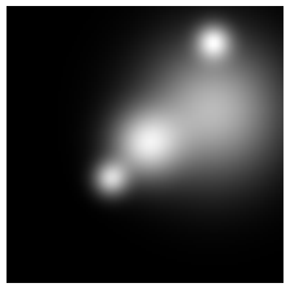
- More complex objects can be created by blending (superposition) of several potential fields
  - Simplest blending is (weighted) addition of the potential fields:

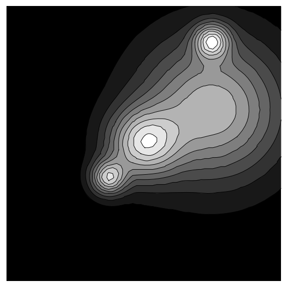
$$P(\mathbf{x}) = \sum_{i=1}^{n} a_i \frac{1}{r_i^2(\mathbf{x})} \ , \qquad r_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_i\|$$

  - The set of points x_i is called the *skeleton*,
    P is the total potential, the a_i determine the influence (= "field's force")
  - Negative influence can "carve out" material (e.g., for making holes)
- Note: the potential field is defined in the whole space


Individual fields



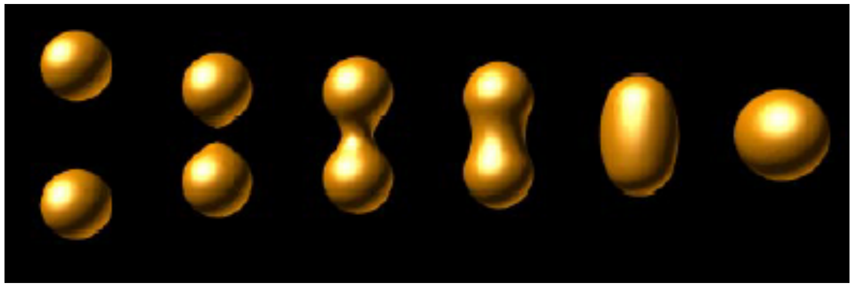


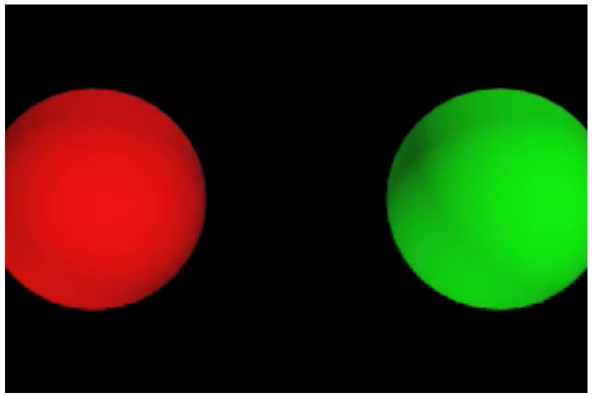

Potential blob shapes

- Ingredients for definition of metaballs:
  distance function, potential function, skeleton points, weights
- In general, a metaballs object is defined as the isosurface

$$\mathcal{F} = \{ \; P(\mathbf{x}) = \tau \mid \mathbf{x} \in \mathbb{R}^3 , \; P(\mathbf{x}) = \sum a_i p(d_i(\mathbf{x})) \; \}$$

  with p = potential function,
  d_i = distance function to i-th skeletal point
- Examples for 2 skeleton points:





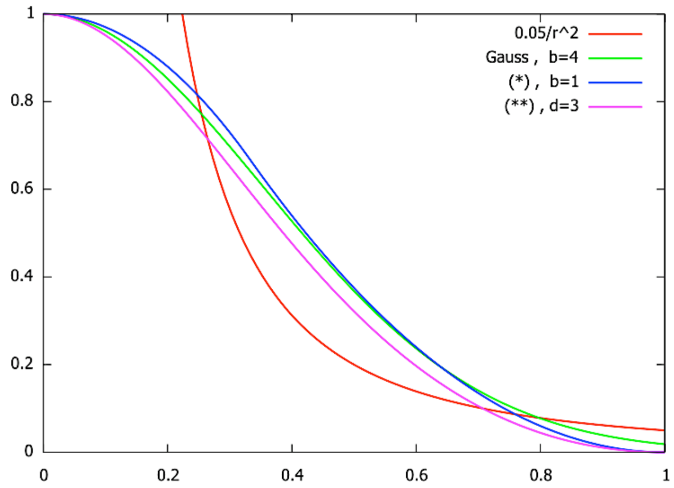
movie

- Other potential functions:

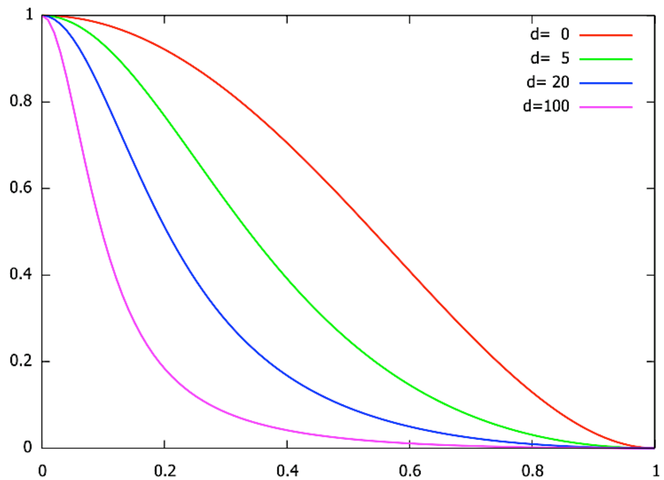
$$p_i(r) = e^{-br^2}$$

$$p(r) = \begin{cases} 1 - 3\frac{r^2}{b^2} & , \; r \leq \frac{1}{3}b \\ \frac{3}{2}(1 - \frac{r}{b})^2 & , \; \frac{1}{3}b \leq r \leq b \\ 0 & , \; r > b \end{cases} \qquad (*)$$

$$p(r) = \begin{cases} \frac{r^4 - 2r^2 + 1}{1 + dr^2} & , \; r \leq 1 \\ 0 & , \; r > 1 \end{cases} \qquad (**)$$

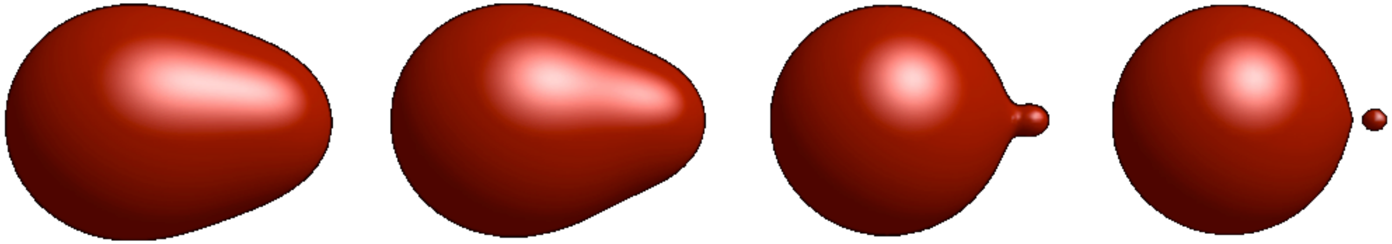


Vergleich



(**) mit verschiedenen Parametern

- Effect of the variation of the parameter *d* :



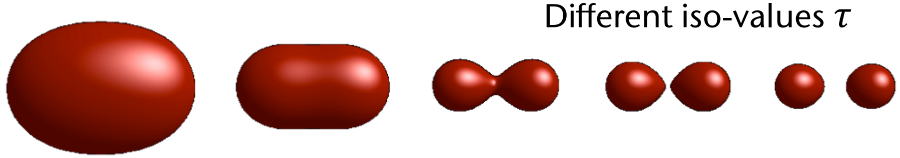
Potential fct is (**), *d* is fixed for the left skeleton point, *d* = 10 … 2000 for the right skeleton point

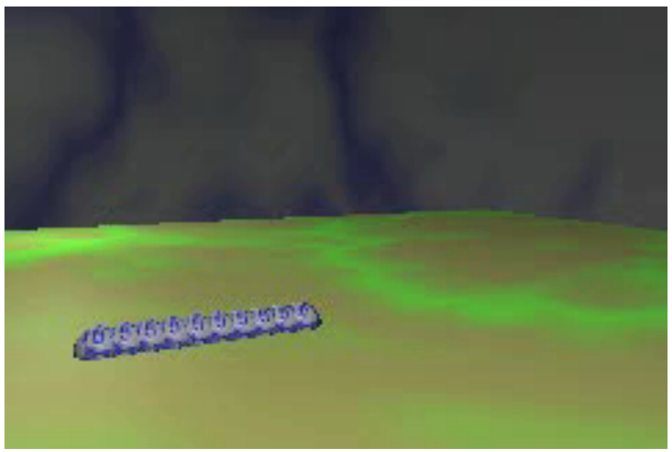
- Many names for this kind of modeling methodology: "metaballs", "soft objects", "blobs", "blobby modeling", "implicit modeling", …

Deformable Models

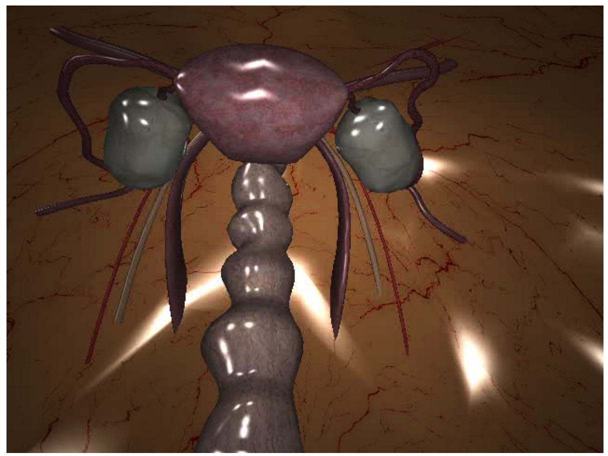
- With implicit modeling (metaballs), it is easy to create and animate deformable "blob-like" objects:
  - Animate (move) the skeleton points
  - Modify parameters a_i, d, …
  - Modify the iso-value τ

Different iso-values τ

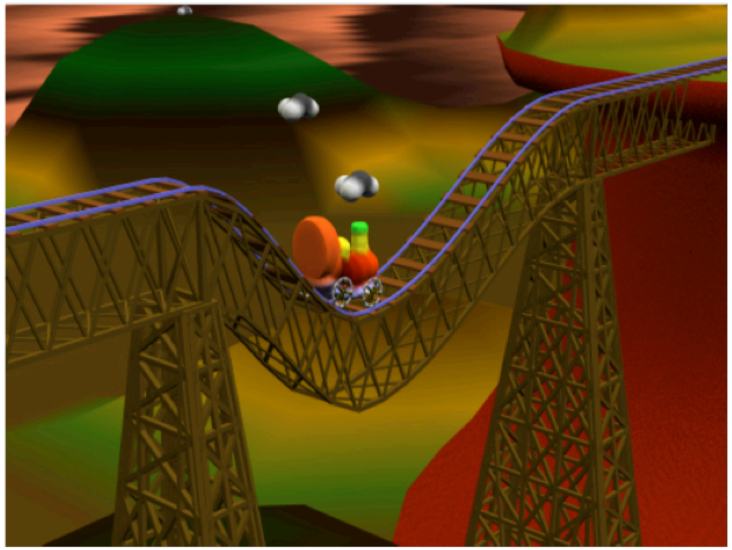




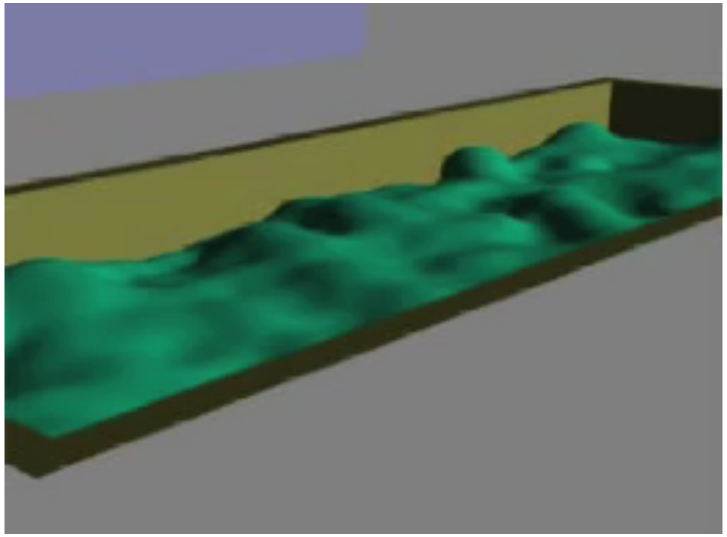
Brian Wyvill
http://pages.cpsc.ucalgary.ca/~blob/animations.html



Frédéric Triquet
http://www2.lifl.fr/~triquet/implicit/video/



"The Great Train Rubbery" — Siggraph 1986



"Soft"

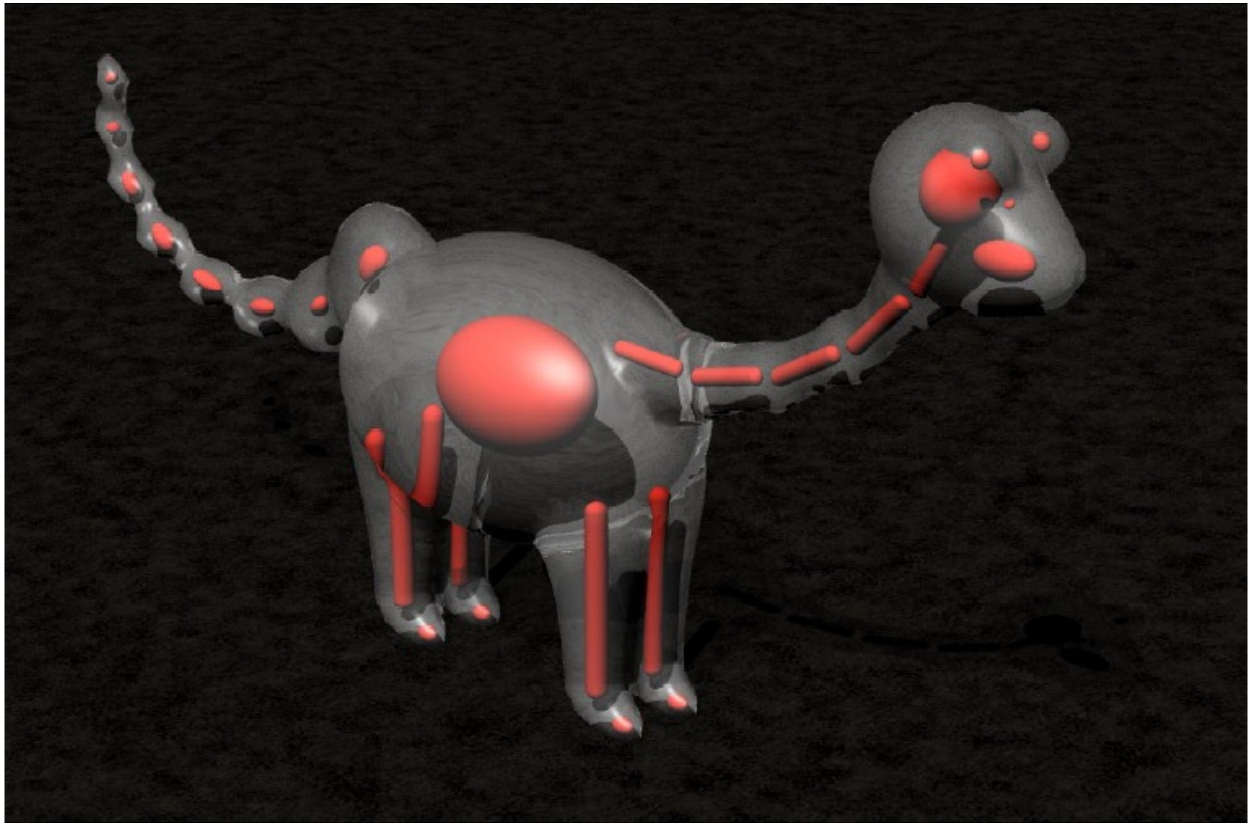
"The Wyvill Brothers"

Geoff

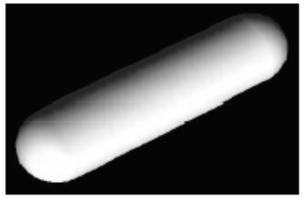
Brian

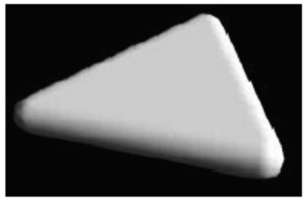
Generalization / Variants

- Points are the simplest kind of primitive for metaballs skeletons; analogously, we can use lines, polygons, ellipsoids, etc.:



Examples of other primitives:



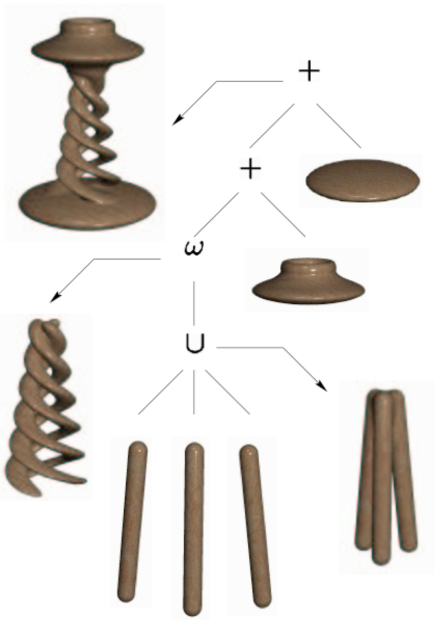


- Other blending functions:

$$P_{\cup}(\mathbf{x}) = \max\{p_1(\mathbf{x}), p_2(\mathbf{x})\}$$

$$P_{\cap}(\mathbf{x}) = \min\{p_1(\mathbf{x}), p_2(\mathbf{x})\}$$

- A tree of "blending" operations (similar to CSG) — the "BlobTree":



Remarks on Implicit Modeling

- One can achieve some nice effects very easily
- The technique did not get traction in the tool set of animation industries and CAD, because there is too much "black magic" involved in achieving a particular effect [says Geoff Wyvill, too]
- For special kinds of deformable objects, it can be very useful, e.g., for fluids